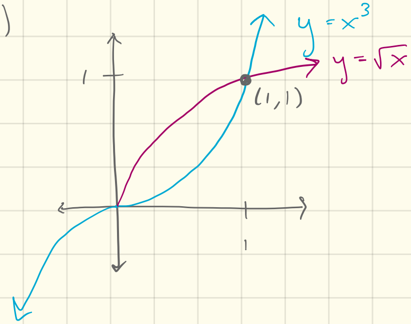
Portfolio 2

Sample

1. Determine the center of mass for the region bounded by the curves $y=\sqrt{x}$ and $y=x^3$.

Solution: First, let's draw the region described. Note that the curves intersect at (0, 0) and (1, 1)



Now, we will compute the area of the region:

$$A = \int_0^1 \sqrt{x} - x^3 \, dx$$

$$= \int_0^1 x^{1/2} - x^3 \, dx$$

$$= \frac{x^{3/2}}{3/2} - \frac{x^4}{4} \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{4}$$

$$= \frac{5}{12}.$$

Salmon 2

Using $f(x) = \sqrt{x} - x^3$, we can use the formulas

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx, \quad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}(f(x))^2\,dx$$

to compute the center of mass. Thus,

$$\bar{x} = \frac{1}{5/12}\int_0^1 x(\sqrt{x} - x^3)\,dx$$

$$= \frac{12}{5}\int_0^1 x^{3/2} - x^4\,dx$$

$$= \frac{12}{5}\left(\frac{x^{5/2}}{5/2} - \frac{x^5}{5}\right)\Bigg|_0^1$$

$$= \frac{12}{5}\left(\frac{2}{5} - \frac{1}{5}\right)$$

$$= \frac{12}{25}$$

$$\bar{y} = \frac{1}{5/12}\int_0^1 \frac{1}{2}\left(\sqrt{x} - x^3\right)^2 dx$$

$$= \frac{6}{5}\int_0^1 \left(x^{1/2} - x^3\right)^2 dx$$

$$= \frac{6}{5}\int_0^1 x - 2x^{7/2} + x^6\,dx$$

$$= \frac{6}{5}\left(\frac{x^2}{2} - \frac{2x^{9/2}}{9/2} + \frac{x^7}{7}\right)\Bigg|_0^1$$

$$= \frac{6}{5}\left(\frac{1}{2} - \frac{4}{9} + \frac{1}{7}\right)$$

$$= \frac{5}{21}$$

Therefore, the center of mass of the given region is $\left(\frac{12}{25} , \frac{5}{21} \right)$.

2. Determine if the sequence $\{a_n\}$ defined below converges or diverges. If it converges, find the limit.

$$a_n = \frac{3^{n+2}}{5^n}.$$

Solution: Consider

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^{n+2}}{5^n}$$

$$= \lim_{n\to\infty} 3^2\left(\frac{3}{5}\right)^n$$

$$= 0 \quad \left(\text{since } \left|\frac{3}{5}\right| < 1\right).$$

Thus, $\{a_n\}$ converges to 0.

3. For which values of x does the series

$$\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^n}$$

converge? For those values, find the sum of the series.

Solution: Note that $\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^n}$ is a geometric series with common ratio $r = \frac{x+4}{3}$. Hence, the series will converge for $|r| < 1$. That is,

$$\left| \frac{x+4}{3} \right| < 1$$

$$-1 < \frac{x+4}{3} < 1$$

$$-3 < x+4 < 3$$

$$-7 < x < -1.$$

Therefore, $\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^n}$ will converge for x satisfying $-7 < x < -1$.

Now, suppose x is in $(-7, -1)$. The first term of the series is $\frac{(x+4)^0}{3^0} = 1$. so the sum of the series is

$$\sum_{n=0}^{\infty} \frac{(x+4)^n}{3^n} = \frac{1}{1 - \left(\frac{x+4}{3}\right)}.$$

4. Use the integral test to determine if the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

converges or diverges.

Solution: Consider the function $f(x) = \frac{1}{x \ln(x)}$ on the interval $[2, \infty)$. Note that for any x in $[2, \infty)$ or any positive integer n, both inequalities below hold:

$$\frac{1}{x \ln(x)} > 0 \quad \text{and} \quad \frac{1}{n \ln(n)} > 0.$$

Further, f is continuous on $[2, \infty)$ and decreasing on $, \infty)$ (as x and $\ln(x)$ are both increasing functions).

Now, consider the improper integral $\int_2^{\infty} \frac{1}{x \ln(x)} dx$.

By definiton, we have

$$\int_2^{\infty} \frac{1}{x \ln(x)} dx = \lim_{t \to \infty} \int_2^t \frac{1}{x \ln(x)} dx \qquad \boxed{\begin{aligned} u &= \ln(x) \\ du &= \frac{1}{x} dx \end{aligned}}$$

$$= \lim_{t \to \infty} \int_2^t \frac{1}{u} du$$

$$= \lim_{t \to \infty} [\ln(u)] \Big|_2^t$$

$$= \lim_{t \to \infty} \ln(t)$$

$$= \infty.$$

Since $\int_2^\infty \frac{1}{x\ln(x)}\,dx$ diverges, the series

$\sum_{n=2}^\infty \frac{1}{n\ln(n)}$ diverges by the integral test.

5. Determine if $\sum_{n=1}^{\infty} \frac{1}{e^n+n^2}$ converges or diverges.

Solution: Let $a_n = \frac{1}{e^n+n^2}$ and let $b_n = \frac{1}{n^2}$.

Note that for all positive integers n, a_n and b_n are positive. Further, we have that

$$e^n + n^2 > n^2$$

So $a_n = \frac{1}{e^n+n^2} < \frac{1}{n^2} = b_n$.

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test $(p=2>1)$.

Hence, by the direct comparison test, $\sum_{n=1}^{\infty} \frac{1}{e^n+n^2}$ also converges.

6. Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-n+5}{n^3-3n+6}$ converges or diverges.

Solution: Define $a_n = \dfrac{n^2-n+5}{n^3-3n+6}$ and $b_n = \dfrac{1}{n}$.

Note that a_n and b_n are positive for all positive integers n. Consider

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\;\frac{n^2-n+5}{n^3-3n+6}\;}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n^2-n+5}{n^3-3n+6} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{n^3-n+5}{n^3-3n+6}$$

$$= 1 > 0$$

Also, $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test ($p=1$).

Since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$, by the limit comparison test,

$\displaystyle\sum_{n=1}^{\infty} \frac{n^2-n+5}{n^3-3n+6}$ diverges.